9. Let A_{ij} be the cofactor of the element a_{ij} in the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $\det(A) = 5$. The value of the expression $a_{11}A_{11} + a_{12}A_{12} + a_{21}A_{21} + a_{22}A_{22}$ then is equal to

A. 0

B. 5

C. 10

D. 15

E. undetermined by the information given above

$\det A = a_{11}A_{11} + a_{12}A_{12}$ = expansion along row 1

$\det A = a_{21}A_{21} + a_{22}A_{22}$ = expansion along row 2

So this is $2 \times \det A = 10$

10. Let $A = \begin{bmatrix} 1 & k \\ k & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The values of k for which the system $Ax = b$ has a unique solution are

A. $k \neq \sqrt{6}$

B. $k \neq \{\sqrt{3}, -\sqrt{3}\}$

C. $k = \{\sqrt{2}, -\sqrt{2}\}$

D. All real numbers

E. None of the above

The system has a unique solution only if the matrix has an inverse

$\det \begin{bmatrix} 1 & k \\ k & 3 \end{bmatrix} \neq 0$ $3 - k^2 \neq 0$

$k \neq \sqrt{3}$, $k \neq -\sqrt{3}$

Newton's Law of Cooling: M = temperature of the ambient, $T(t)$ = temperature of the object

$\frac{dT}{dt} = k(M-T)$, $k>0$; $\frac{1}{M-T}\,dT = k\,dt$

$-\ln|M-T(t)| = kt + C$ $\quad |M-T(t)| = e^{-kt}\cdot e^{c}$

$\boxed{M-T(t) = (M-T(0))\,e^{-kt}}$

2. A ranger in a national park found at noon the body of a wild boar killed by a poacher. To convict a suspected poacher, one needs to know the time when the boar was killed. To determine this time, the temperature of the body was measured twice: at noon it was 60° F and at 1 p. m. it was 55° F. The air temperature on that day was 50° F and it did not change since 8 a.m. The normal body temperature of the boar when it is alive is 90° F. When was the boar killed?

A. 6 a.m.

B. 10 a.m.

C. 10:30 a.m.

D. 11 a.m.

E. 11:30 a.m.

In this case $M = 50$

Set $t = 12$ p.m as $t=0$. We start counting time at $t=0$, which is 12:00 noon.

$T(0) = 60$, $T(1) = 55$.

$50 - T(t) = (50-60)\,e^{-kt}$; $50 - T(t) = -10e^{-kt}$

$\boxed{T(t) = 50 + 10e^{-kt}}$ $\quad T(1) = 55$

So $55 = 50 + 10e^{-k}$

$10e^{-k} = 5$

$\boxed{e^{-k} = 1/2}$

So $\boxed{T(t) = 50 + 10\left(\frac{1}{2}\right)^{t}}$

The temperature of the body of the boar as killed was 90 at the time the boar

$90 = 50 + 10\left(\frac{1}{2}\right)^{t}$ $\quad 40 = 10\left(\frac{1}{2}\right)^{t}$ $\quad \left(\frac{1}{2}\right)^{t} = 4$

$\boxed{t = -2}$

The boar was killed two hours before

11. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -3 \\ 9 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -2 \\ 4 \\ 2 \\ 8 \end{bmatrix}$. The dimension of the space $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is then equal to

A. 1

B. 2

C. 3

D. 4

E. 5.

The dimension of the span of the four vectors is equal to the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ -1 & 1 & -3 & 4 \\ 1 & 7 & 9 & 2 \\ -1 & 3 & 1 & 8 \end{bmatrix} \xrightarrow[R_1+R_4]{R_1+R_2,\ -R_1+R_3} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 4 & 2 & 2 \\ 0 & 4 & 4 & 4 \\ 0 & 6 & 6 & 6 \end{bmatrix}$$

$$\xrightarrow[-R_3/4+R_4/6]{R_2/2,\ R_3/4} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Swap } R_2 \to R_3} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{Rank} = 3}$$

$M = 6x^2y^2 + 4e^x - 2y \sin 2x$; $\frac{\partial M}{\partial y} = 12yx^2 - 2\sin 2x$

$N = 4x^3y + \cos 2x$, $\frac{\partial N}{\partial x} = 12x^2y - 2\sin 2x$

8. Find an implicit solution of the initial value problem:

$$\begin{cases} (6x^2y^2 + 4e^x - 2y\sin 2x) + (4x^3y + \cos 2x)\dfrac{dy}{dx} = 0, \\ y(0) = 1 \end{cases}$$

A. $2x^3y^2 - y\cos 2x + 4e^x = 3$

B. $x^3y^2 + y\cos 2x + 4ye^x = 0$

C. $\boxed{2x^3y^2 + y\cos 2x + 4e^x = 5}$

D. $x^3y^2 + y^2\cos 2x + 4e^x = 5$

E. $2x^3y^2 + y\cos 2x - 4ye^x = -3$

$\frac{\partial F}{\partial x} = 6x^2y^2 + 4e^x - 2y\sin 2x$

$F(x,y) = 2x^3y^2 + 4e^x + y\cos 2x + f(y)$

$\frac{\partial F}{\partial y} = 4x^3y + \cos 2x + f'(y) =$

$4x^3y + \cos 2x$

so $f'(y) = 0$

$F(x,y) = 2x^3y^2 + 4e^x + y\cos 2x = C$

when $x=0$, $y=1$

$F(0,1) = 4 + 1 = 5$

$$\boxed{2x^3y^2 + 4e^x + y\cos 2x = 5}$$

3. The general solution of $xy' - y = x^2e^x$ is

A. $\boxed{y = xe^x + cx}$

B. $y = x^2e^x - xe^x + cx$

C. $y = xe^x - cx^2$

D. $y = x^2e^x + xe^x + cx$

E. None of the above

$$y' - \frac{y}{x} = xe^x$$

$$x\frac{d}{dx}(x^{-1}y) = xe^x$$

$$\frac{d}{dx}(x^{-1}y) = e^x$$

$$x^{-1}y = e^x + c$$

$$\boxed{y(x) = xe^x + cx.}$$

6. The solution of

$$y' + \frac{y}{x} = \frac{2}{x^2 y}, \qquad x \neq 0$$

is given by

A. $x^2y^2 + 4xy = C$

B. $x^2y^2 + 4x = C$

C. $xy^2 - 2x = C$

D. $x^2y^2 - 4x = C$

E. $xy^2 - 4x = C$

multiply the equation by y:

$v = y^2$

$\frac{dv}{dx} = 2yy'$

$yy' + \frac{1}{x}y^2 = \frac{2}{x^2}$

$\frac{1}{2}v' + \frac{1}{x}v = \frac{2}{x^2}$

$v' + \frac{2}{x}v = \frac{4}{x^2}$

$x^{-2}\frac{d}{dx}(x^2 v) = 4x^{-2}$

$\frac{d}{dx}(x^2 v) = 4$

$x^2 v = 4x + c$

but $v = y^2$.

$\boxed{x^2y^2 - 4x = C}$

3. Find the explicit solution of the initial value problem

$$y' = \frac{xy^2}{x^2+1}, \quad y(0) = 3.$$

$$\frac{dy}{y^2} = \frac{x}{x^2+1}\,dx$$

A. $\frac{1}{2}(6 + \ln(1 + x^2))$

B. $\dfrac{6}{2 - 3\ln(1 + x^2)}$

C. $\dfrac{6}{2 + 3\ln(1 + x^2)}$

D. $\frac{1}{2}(6 - 3\ln(1 + x^2))$

E. $\frac{1}{3}(9 - 2\ln(1 + x^2))$

$$\int \frac{dy}{y^2} = \int \frac{x}{x^2+1}\,dx$$

$$-\frac{1}{y} = \frac{1}{2}\ln(x^2+1) + C.$$

when $x=0$; $-\frac{1}{3} = C.$

$$-\frac{1}{y} = -\frac{1}{3} + \frac{1}{2}\ln(x^2+1)$$

$$\frac{1}{y} = \frac{1}{3} - \frac{1}{2}\ln(1+x^2)$$

$$\frac{1}{y} = \frac{2 - 3\ln(1+x^2)}{6}$$

$$y = \frac{6}{2 - 3\ln(1+x^2)}$$

The homogeneous equation is $y'' + 4y = 0$ and its general solution is $y_h(x) = c_1 \cos 2x + c_2 \sin 2x$.

One needs to find a particular solution. Since $\sin(2t)$ is a solution of the homogeneous eq. one needs to look for $y_p(x) = A e^{2t} + Bt \sin 2t + C t \cos 2t$.

8. Which of the following is the general solution to $y'' + 4y = e^{2t} + 12\sin(2t)$?

Correct

A. $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} - 3t\cos(2t)$

Not this one

B. $\boxed{y(t) = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4}t\,e^{2t} - 3t\cos(2t)}$

C. $y(t) = c_1 + c_2 e^{-4t} + \frac{1}{12}te^{2t} - 3t\cos(2t)$

D. $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} + 3\sin(2t)$

E. None of the above.

we look for A:

$y_{1p} = A e^{2t}$

$y'_{p1} = 2Ae^{2t}$; $y''_{p1} = 4Ae^{2t}$

$y''_{p1} + 4y_{p1} = 8Ae^{2t} = e^{2t}$ $A = \frac{1}{8}$

$y_{p2} = Bt \sin 2t + C t \cos 2t$, $y'_{p2} = B \sin 2t + C \cos 2t + 2Bt \cos 2t - 2Ct \sin 2t$

$y''_p = 4B \cos 2t - 4C \sin 2t - 4Bt \sin 2t - 4Ct \cos 2t.$

$y''_p + 4y = 4B \cos 2t - 4C \sin 2t = 12 \sin 2t$ $-4C = 12$ $C = -3$

So $y_p(t) = \frac{1}{8} e^{2t} - 3t \cos(2t)$

$y = e^{rt}$

$r^4 - r^2 = 0$

$r^2(r^2-1) = 0$

$r = 0$ multip 2

$r = 1,\ r = -1$

10. A particular solution of the equation

$$y^{(4)} - y'' = 2\sin t - 3e^{-t} + 4t$$

is of the form

A. $y_p(t) = A\cos t + B\sin t + Cte^{-t} + t^2(Dt+E)$

B. $y_p(t) = t(A\cos t + B\sin t) + Cte^{-t} + t^2(Dt+E)$

C. $y_p(t) = A\cos t + B\sin t + Ce^{-t} + t^2(Dt+E)$

D. $y_p(t) = t(A\cos t + B\sin t) + Ce^{-t} + t^2(Dt+E)$

E. $y_p(t) = t(A\cos t + B\sin t) + Cte^{-t} + t(Dt+E)$

y_p corresponding to $\sin 2t$

$y_p = A\sin 2t + B\cos 2t$

y_p corresponding to $4t$ $\quad y_p = t^2(Ct+D)$.

y_p corresponding to $-3e^{-t}$.

$y_p = Ete^{-t}$

$y_p = A\sin 2t + B\cos 2t + t^2(Ct+D) + Ete^{-t}$.

9. According to the method of undetermined coefficients, what is the proper form of a particular solution Y to the following differential equation?

$$y^{(4)} - 4y'' = 24t^2 - 4 - 3te^t.$$

$y = e^{rt}$

$r^4 - r^2 = r^2(r^2-1)$

$r=0$ multiplicity 2

$r=1$, $r=-1$

A. $Y(t) = At^2 + Bte^t$.

B. $Y(t) = At^2 + Bt + C + Dte^t + Ee^t$.

C. $Y(t) = At^3 + Bt^2 + Ct + D + Ete^t + Fe^t$.

$\boxed{\text{D. } Y(t) = At^4 + Bt^3 + Ct^2 + Dte^t + Ee^t.}$

E. None of the above.

y_p corresponding to : $24t^2 - 4$ $\quad y_{p1} = t^2(A + Bt + Ct^2)$

y_p corresponding to $-3te^t$ $\quad y_p = t(tD + E)e^t$

5. How many asymptotically unstable equilibrium solution(s) does the following differential equation have?

$$y' = (y^2 + 1)(y^2 - 1)(y + 2).$$

The real roots are $y=-2$, $y=-1$, $y=1$

A. 0,

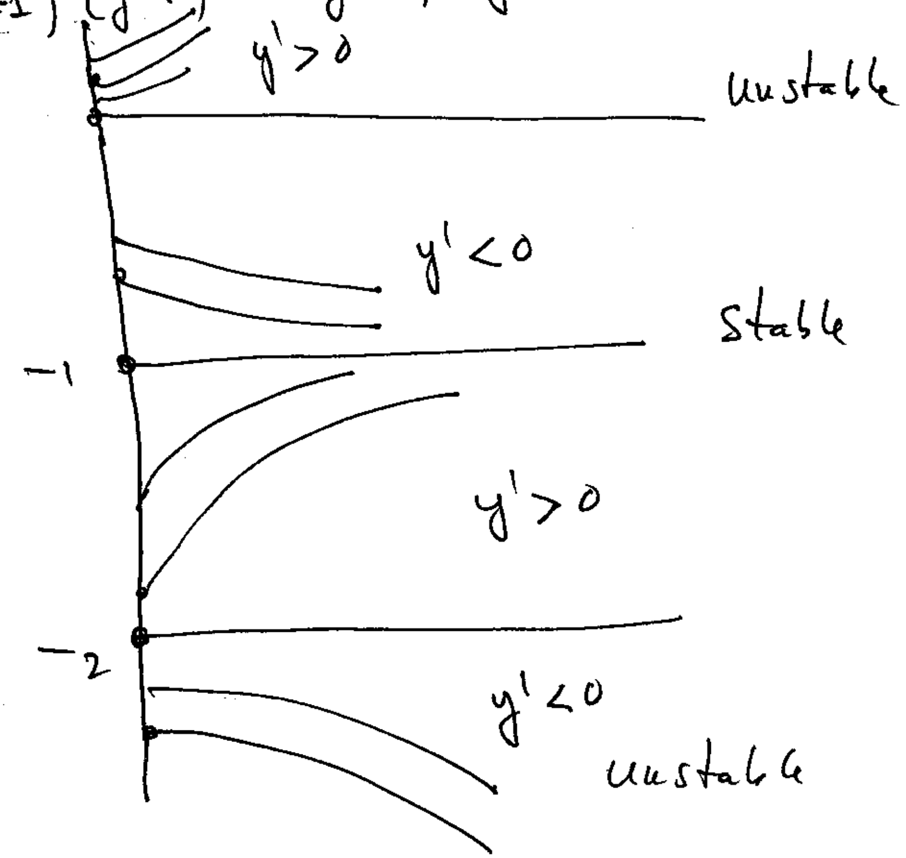
B. 1,

C. 2,

D. 3,

E. None of the above.

$y' = (y^2+1)(y+2)(y+1)(y-1)$



If $y<-2$ $\quad y'<0$

If $-2<y<-1$ $\quad y'>0$

If $-1<y<1$

3. Find the solution $y(x)$ to

$$\frac{dy}{dx} = e^{-\frac{y}{x}} + \frac{y}{x},$$
$$y(e) = 0.$$

A. $y(x) = \ln(\ln(x))$

B. $y(x) = x\ln(\ln(x))$

C. $y(x) = \ln(\ln(x) + 1) - \ln(2)$

D. $y(x) = x\ln(x) - e$

E. $y(x) = x\ln(x) - x$

Set $y/x = V$

$y = xV$

$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$V + x\frac{dV}{dx} = V + e^{-V}.$$

$$x\frac{dV}{dx} = e^{-V}$$

$$e^{V}\,dV = \frac{dx}{x}.$$

$$e^{V} = \ln x + c$$

When $x = e$; $y(e) = 0$ So $V(e) = 0$

So $c = 0$.

$e^{V} = \ln x$ $\quad V = \ln(\ln x)$

$$\boxed{y = x\ln(\ln x)}$$

14. Using the method of Undetermined Coefficients, which of the following is the correct form of a particular solution y_p to the nonhomogeneous differential equation

$$y^{(4)} - y = xe^x + 3\cos x - 4x \text{ ?}$$

A. $y_p = x(A + Bx)e^x + (C\cos x + D\sin x) + Ex$

B. $y_p = (A + Bx)e^x + (C\cos x + D\sin x) + E + Fx$

C. $y_p = (A + Bx)e^x + x(C\cos x + D\sin x) + x(E + Fx)$

D. $y_p = x(A + Bx)e^x + x(C\cos x) - Dx$

E. $y_p = x(A + Bx)e^x + x(C\cos x + D\sin x) + E + Fx$

9. Find the general solutions to $y'' + 6y' + 10y = 0$

$y = e^{rx}$

$r^2 + 6r + 10 = 0$

$r = \frac{-6 \pm \sqrt{36 - 40}}{2}$

$r = \frac{-6 \pm \sqrt{-4}}{2}$

$r = -3 \pm i$

$y(t) = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t$

A. $c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$

B. $c_1 e^{-3t}\cos(t) + c_2 e^{-3t}\sin(t)$

C. $c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$

D. $c_1 e^{2t}\cos(t) + c_2 e^{2t}\sin(t)$

E. $c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$

Eigenvalues! $\det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix} = -(3+\lambda)(1-\lambda)+4$
$= -3+3\lambda-\lambda+\lambda^2+4$
$= \lambda^2+2\lambda+1 = (\lambda+1)^2.$

Solution. $X_1(t) = e^{-t} v_1$

$X_2(t) = e^{-t}(t v_1 + v_2)$

V_2 is any vector.
such that $(A+I)V_2 = V_1 \neq 0$

$A+I = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$

We pick $V_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$V_1 = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

7. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

A. $c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t}\right]$,

B. $c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}e^{-t}\right]$,

C. $c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t}\right]$,

D. $c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-t} + c_2\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t}\right]$,

E. $c_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{t} + c_2\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t}\right]$.

So $X_1(t) = e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$X_2(t) = e^{-t}\left(t\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

2. Find the number of **stable** critical points for the autonomous equation

$$\frac{dx}{dt} = x(x-1)^2(x+3)(x^2-4) = F(x) = (x+3)(x+2)\,x\,(x-1)^2(x-2)$$

A. 1

B. 2

C. 3

D. 4

E. 0

the critical points are the zeros of $F(x)$: $-3, -2, 0, 1, 2$

When $x < -3$, $F(x) > 0$

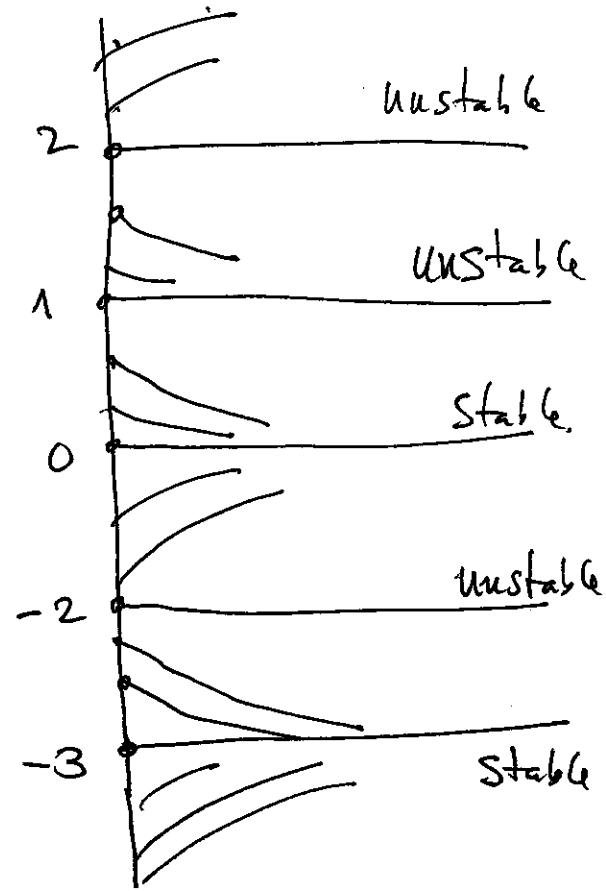
$-3 < x < -2$, $F(x) < 0$

$-2 < x < 0$, $F(x) > 0$

$0 < x < 1$, $F(x) < 0$

$1 < x < 2$, $F(x) < 0$

$2 < x$ $F(x) > 0$



16. One eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ -1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is $\lambda = 3$. A basis for the corresponding eigenspace is

A. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$(A - 3I)\,V = 0$ $\quad \begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-v_1 + v_2 + v_3 = 0 \qquad v_1 = v_2 + v_3$

$$V = \begin{bmatrix} v_2 + v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

One basis is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

but if we add the two vectors

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is another basis

17. If $y(x)$ is a solution of $y'' - 2y' + y = 0$ satisfying $y(0) = 1$ and $y'(0) = -1$, then $y(\frac{1}{2}) =$

A. 0

B. $-e^{\frac{1}{2}}$

C. $e^{\frac{1}{2}}$

D. $2e^{\frac{1}{2}}$

E. $-3e^{\frac{1}{2}}$

$y = e^{rx}$; $r^2 - 2r + 1 = 0$ $(r-1)^2 = 0$

$y(x) = Ae^x + Bxe^x$

$y(0) = A = 1$

$y'(x) = Ae^x + Be^x + Bxe^x$

$y'(0) = A + B = -1$ $B = -2$

$y(x) = e^x - 2xe^x$

$y(\frac{1}{2}) = e^{\frac{1}{2}}(1 - 2 \cdot \frac{1}{2}) = 0$

Step 1: Find eigenvalues. $\det\begin{bmatrix}3-\lambda & 5\\ -1 & -1-\lambda\end{bmatrix} = (3-\lambda)(-1-\lambda)+5 = \lambda^2-2\lambda+2=0$

$\lambda = \frac{2\pm\sqrt{-4}}{2} = 1\pm i$

Step 2: Find eigenvectors:

$\lambda = 1+i$, $(A-\lambda I)v = \begin{bmatrix}(2-i) & 5\\ -1 & -(2+i)\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$

$v_1+(2+i)v_2=0 \qquad V_1 = -(2+i)V_2$

$V = \begin{bmatrix}-2-i\\ 1\end{bmatrix}$

24. The general solution of $\mathbf{x}' = \begin{bmatrix}3 & 5\\ -1 & -1\end{bmatrix}\mathbf{x}$ has the form

A. $c_1e^t\begin{bmatrix}2\cos t-\sin t\\ -\cos t\end{bmatrix}+c_2e^t\begin{bmatrix}\cos t+2\sin t\\ -\sin t\end{bmatrix}$

B. $c_1e^t\begin{bmatrix}\cos t\\ 2\cos t-\sin t\end{bmatrix}+c_2e^t\begin{bmatrix}-\sin t\\ \cos t-2\sin t\end{bmatrix}$

C. $c_1e^t\begin{bmatrix}2\cos t+\sin t\\ -\cos t\end{bmatrix}+c_2e^t\begin{bmatrix}\cos t+2\sin t\\ \sin t\end{bmatrix}$

D. $c_1e^t\begin{bmatrix}\cos t-2\sin t\\ \sin t\end{bmatrix}+c_2e^t\begin{bmatrix}2\cos t+\sin t\\ -\cos t\end{bmatrix}$

E. None of the above.

$X(t) = e^{(1+i)t}\begin{bmatrix}-2-i\\ 1\end{bmatrix}$

The real and imaginary parts will give two L.I real solution.

$X(t) = e^t(\cos t + i\sin t)\begin{bmatrix}-2-i\\ 1\end{bmatrix} = e^t\begin{bmatrix}-2\cos t+\sin t - i(\cos t+2\sin t)\\ \cos t + i\sin t\end{bmatrix}$

$= e^t\begin{bmatrix}-2\cos t+\sin t\\ \cos t\end{bmatrix} + ie^t\begin{bmatrix}-\cos t - 2\sin t\\ \sin t.\end{bmatrix}$

The method of variations of parameters gives that

$$u_1' = -\frac{1}{W(y_1,y_2)} y_2 f ; \qquad u_2' = \frac{1}{W(y_1,y_2)} y_1 f.$$

$$W(y_1,y_2) = \begin{bmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{bmatrix} = -4.$$

7. If $y = u_1 y_1 + u_2 y_2$ where $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ is a particular solution of

$$y'' - 4y = 4 \tan x$$

then u_1 and u_2 are determined by

A. $u_1' = e^{-2x} \tan x \quad u_2' = -e^{2x} \tan x$

B. $u' = -e^{-2x} \sec^2 x \quad u_2' = e^{2x} \sec^2 x$

C. $u_1' = -2 \sin 2x \tan x \quad u_2' = 2 \cos 2x \tan x$

D. $u_1' = 2 \sin 2x \tan x \quad u_2' = -2 \cos 2x \tan x$

E. $u_1' = \tan x \quad u_2' = 0$

So $u_1'(x) = \frac{1}{4} e^{-2x} (4 \tan x)$

$$\boxed{u_1'(x) = e^{-2x} \tan x}$$

$$u_2'(x) = -\frac{1}{4} e^{2x} \cdot (4 \tan x)$$

$$u_1'(x) = - e^{2x} \tan x.$$

10. **Find all values of a such that the following system of equations has exactly one solution.**

$$\begin{cases} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (a^2 - 5)z = a \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

If $a^2-4 \neq 0$ the system has only one Solution

A. $\boxed{a \neq 2 \text{ and } a \neq -2}$

B. $a = 2$ or $a = -2$

C. $a = -2$

D. $a = \pm\sqrt{5}$

E. $a \neq 0$

If $a^2 = 4$ we have two possibilities

$a = 2$: In this case we have infinitely many Solutions

$a = -2$ We have no Solution.

17. Determine all values of k such that the vectors $(1,-1,0)$, $(1,2,2)$, $(0,3,k)$ are a basis for $\mathbb{R}^3$.

$$\det\begin{bmatrix}1&1&0\\-1&2&3\\0&2&k\end{bmatrix} = \det\begin{bmatrix}\cancel{1}&\cancel{1}&\cancel{0}\\ \cancel{0}&3&3\\ \cancel{0}&2&k\end{bmatrix}$$

$$= 3k-6 \neq 0 \qquad k \neq 2$$

A. $k=1$

B. $k=2$

C. $\boxed{k\neq 2}$

D. $k\neq 1$

E. $k\neq 3$

11. Let $y(x)$ be the solution of the initial value problem

$$y'' + y = x,$$
$$y(0) = 1, \quad y'(0) = 2.$$

Then $y(\pi)$ is equal to

A. $y(\pi) = \pi$

B. $y(\pi) = 2\pi$

C. $y(\pi) = \pi - 1$

D. $y(\pi) = 2\pi + 1$

E. $y(\pi) = \pi + 1$

take $y_p = Ax + B$

$y_p'' = 0$ $\quad A = 1. \quad , B = 0$

$y(x) = A \cos x + B \sin x + x.$

$y(0) = A = 1$

$y'(0) = B + 1 = 2 \quad B = 1$

$y(x) = \cos x + \sin x + x.$

$y(\pi) = \pi - 1$

(*) $y(x) = 1 + e^x - \sin x + \cos x$

$y(\pi) = e^{\pi}$

① Solve the homogeneous eq. $y = e^{rx}$ $\quad r^2 - r = 0$

$r = 0, r = 1$

$y_h = C_1 + C_2 e^x$

12. Let $y(x)$ satisfy

$$y'' - y' = 2\sin(x),$$
$$y(0) = 3, \quad y'(0) = 0.$$

Then $y(\pi)$ is equal to:

A. $e^{\pi} + 10$

B. $e^{\pi} + 1$

C. $e^{\pi} + 2$

D. $e^{\pi} + 3$

E. e^{π}

② Find a particular solution.

$y_p(x) = A \sin x + B \cos x$

$y_p' = A \cos x - B \sin x$

$y_p'' = -A \sin x - B \cos x$

$y_p'' - y_p' = -A \sin x - B\cos x - A \cos x + B \sin x = \cos x(-A-B) + \sin x (B-A)$

We want $A + B = 0$; $B - A = 2$ $\quad B = 1, A = -1$

$y(x) = c_1 + c_2 e^x - \sin x + \cos x$

$y'(x) = c_2 e^x - \cos x - \sin x$

$y(0) = c_1 + c_2 + 1 = 3$

$y'(0) = c_2 - 1 = 0$

$c_2 = 1, c_1 = 1$

Go to the top (*)

12. For the system $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -8 & -7 \end{bmatrix} \mathbf{x}$, the origin is

A. a saddle point

B. a proper node source

C. a center point

D. a spiral sink

E. a spiral source

$$\det(A-\lambda I) = \det \begin{bmatrix} 5-\lambda & 5 \\ -8 & -7-\lambda \end{bmatrix}$$

$$= (5-\lambda)(-7-\lambda) + 40$$

$$= (7+\lambda)(\lambda-5) + 40$$

$$= \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$\lambda = \frac{-2 \pm 4i}{2}$$

$$\boxed{\lambda = -1 \pm 2i}$$

Complex eigenvalues with negative Real part.

15. The solution curves of the system

$$X'(t) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} X(t),$$

where $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, have the following structure near the origin $x_1 = x_2 = 0$:

A. A saddle point

B. A nodal sink

C. A nodal source

D. A center

E. parallel lines.

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$= (\lambda+2)(\lambda-1) - 1$$

$$= \lambda^2 + \lambda - 3 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1+12}}{2}.$$

$$\lambda = \frac{-1+\sqrt{13}}{2}, \quad \lambda = \frac{-1-\sqrt{13}}{2}$$

Real eigenvalues with opposite signs

16. Let $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Find all the values of a so that the origin $x_1 = x_2 = 0$

$$X'(t) = \begin{bmatrix} a & -27 \\ 3 & 0 \end{bmatrix} X(t)$$

is a spiral source of the system, that is, the solution curves are spirals approaching infinity as $t \to \infty$?

A. $0 < a < 27$

$\boxed{\text{B. } 0 < a < 18}$

C. $-18 < a < 18$

D. $0 > a > -18$.

E. $0 > a > -27$.

The origin is a spiral if the eigenvalues are complex and have positive real part.

$A = \begin{bmatrix} a & -27 \\ 3 & 0 \end{bmatrix}$

$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & -27 \\ 3 & -\lambda \end{bmatrix} = -\lambda(a-\lambda) + 81$

$= \lambda^2 - a\lambda + 81$. the roots are

$\lambda = \frac{1}{2}\left(a \pm \sqrt{a^2 - 4 \times 81}\right)$

So we must have $a > 0$ because the real part is $a/2$ and $a^2 - 4 \times 81 < 0$ So the Eigenvalues are imaginary

$0 < a < \sqrt{4 \times 81} = 18$

Step 1: $\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 6 \\ 1 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4$

$= (\lambda-4)(\lambda+1)$

$\lambda=-1$: $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\quad v_1 = -3v_2 \quad V_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda = 4$ $\begin{bmatrix} -3 & 6 \\ 1 & -2 \end{bmatrix}$ $\quad v_1 = 2v_2 \quad V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$X(t) = c_1 e^{-t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$X(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$-3c_1 + 2c_2 = -1$

$c_1 + c_2 = 2$

$c_1 = c_2 = 1$

17. Let $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ satisfy

$$X'(t) = \begin{bmatrix} 1 & 6 \\ 1 & 2 \end{bmatrix} X(t), \quad X(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The limit $S = \lim_{t\to\infty} \frac{x_2'(t)}{x_1'(t)}$, which is the slope of the curve $X(t)$ as $t \to \infty$, is equal to

A. $S = -3$

B. $S = \frac{1}{2}$

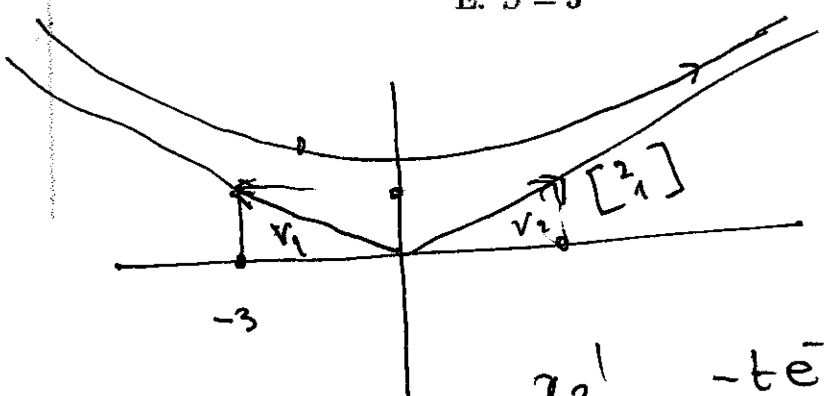
C. $S = -\frac{1}{3}$

D. $S = -\frac{1}{2}$

E. $S = 3$

$X(t) = e^{-t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

as $t \to \infty$ $X(t) \to e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad$ the slope $= \frac{1}{2}$



one can also check:

$x_1 = -3e^{-t} + 2e^{4t}$;

$x_2(t) = e^{-t} + e^{4t}$

$\lim_{t\to\infty} \frac{x_2'}{x_1'} = \frac{4}{8} = \frac{1}{2}$

$\frac{x_2'}{x_1'} = \frac{-te^{-t} + 4e^{4t}}{3e^{-t} + 8e^{4t}}$

15. Which of the following matrices are nondefective:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}?$$

A. A, B

B. A, C

C. B only

D. C only

E. A only

For 2x2 matrices, being nondefective is the same as having two distinct eigenvalues.

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda)$$

Eigenvalues $\lambda = 1$, $\lambda = 3$

$$\det(B-\lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$$

Eigenvalue $\lambda = 1$.

$$\det(C-\lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)$$

$\lambda = 0$, $\lambda = 1$

A and C are nondefective. B is defective.

22. The eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$ are 3 and 5. One of the associated eigenspaces has dimension one. This eigenspace has basis:

$$(A-3I)v = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2v_1 + 2v_3 = 0$ $\quad v_1 = -v_3$.

$$V = \begin{bmatrix} v_1 \\ v_2 \\ -v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigenspace has dimension two.

$$(A-5I)v = \begin{bmatrix} -2 & 0 & 0 \\ -2 & -2 & -2 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 = 0$, $\quad -2v_2 - 2v_3 = 0$ $\quad v_2 = -v_3$

$$V = \begin{bmatrix} 0 \\ v_2 \\ -v_2 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\boxed{\text{A. } \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$

B. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

E. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

8. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 5 & -1 \\ 1 & 6 & -2 \end{bmatrix}$. Let λ_1, λ_2 and λ_3 denote the eigenvalues of A and let E_1, E_2 and E_3 denote the corresponding eigenspaces. Which of the following is correct?

A. $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = -1$, $\dim E_1 = \dim E_2 = \dim E_3 = 1$

B. $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 4$, $\dim E_1 = \dim E_2 = 2$, $\dim E_3 = 1$

C. $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 4$, $\dim E_1 = \dim E_2 = 1$, $\dim E_3 = 1$

D. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 4$, $\dim E_1 = 1$, $\dim E_2 = \dim E_3 = 2$

E. $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 3$, $\dim E_1 = \dim E_2 = 2$, $\dim E_3 = 1$

$$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 0 & 0 \\ 1 & 5-\lambda & -1 \\ 1 & 6 & -2-\lambda \end{bmatrix}$$

$$= -(+1+\lambda) \cdot \left((5-\lambda)(-2-\lambda) + 6 \right)$$

$$= -(1+\lambda)(-10 - 5\lambda + 2\lambda + \lambda^2 + 6)$$

$$= -(1+\lambda)(\lambda^2 - 3\lambda - 4)$$

$$= -(1+\lambda)(\lambda+1)(\lambda-4).$$

$$= -(1+\lambda)^2(\lambda-4).$$

$\underline{\lambda = -1}$: $(A+I)v = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 6 & -1 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$v_1 + 6v_2 - v_3 = 0$ $\quad$ $v_3 = v_1 + 6v_2$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_1 + 6v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = -1$

$\boxed{\dim E_1 = \dim E_2 = 2.}$

$\dim E_3$ must be 1

15. Let A be a 2×2 matrix whose entries are real numbers. If $\lambda = 2 + 3i$ is a complex eigenvalue of A with corresponding complex eigenvector $\mathbf{w} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$, then the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

A. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4 \sin 3t \end{bmatrix}$

B. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t - \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ 4 \sin 3t \end{bmatrix}$

C. $\mathbf{x} = C_1 e^{3t} \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t - \cos 2t \\ 4 \sin 2t \end{bmatrix}$

D. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t + \cos 3t \\ 4 \sin 3t \end{bmatrix}$

E. $\mathbf{x} = C_1 e^{2t} \begin{bmatrix} \cos 3t + \sin 3t \\ 4\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} \sin 3t - \cos 3t \\ -4 \sin 3t \end{bmatrix}$

We need to find the real and imaginary parts of the complex solution

$$X(t) = e^{(2+3i)t} \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$$

$$= e^{2t}\left(\cos(3t) + i\sin(3t)\right)\begin{bmatrix} 1-i \\ 4 \end{bmatrix} =$$

$$= e^{2t}\begin{bmatrix} \cos(3t) - i\cos(3t) + i\sin(3t) + \sin(3t) \\ 4\cos(3t) + 4i\sin(3t) \end{bmatrix} =$$

$$= \underbrace{e^{2t}\begin{bmatrix} \cos(3t) + \sin(3t) \\ 4\cos(3t) \end{bmatrix}}_{X_1(t)} + i\,\underbrace{e^{2t}\begin{bmatrix} \sin(3t) - \cos(3t) \\ 4\sin 3t \end{bmatrix}}_{X_2(t)}$$

Real solution:

General Solution:

$$X(t) = C_1 e^{2t}\begin{bmatrix} \cos(3t) + \sin(3t) \\ 4\cos(3t) \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} \sin(3t) - \cos(3t) \\ 4\sin(3t) \end{bmatrix}$$

17. Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be the solution of the system of differential equations

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}.$$

with the initial data $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Given that the eigenvalues of A and their corresponding eigenvectors of A are $\lambda_1 = -5$, $V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\lambda_2 = 2$, $V_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find $x_1(1)$.

A. $3e^2$

B. $-2e^{-5}$

C. e^{-5}

D. $-3e^{-5}$

E. $e^{-5} + 3e^2$

The general solution is

$$X(t) = c_1 e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$X(0) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & 2 & 3 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -7 & 0 \end{array}\right] \quad \begin{array}{l} c_2 = 0 \\ c_1 = 1 \end{array}$$

$$X(t) = e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So $x_1(t) = e^{-5t}$

$x_2(t) = 3e^{-5t}$.

19. Find all constants b such that the origin is a spiral source of the system

$$X'(t) = \begin{bmatrix} 3 & b \\ 1 & 4 \end{bmatrix} X(t), \quad b \text{ in } \mathbb{R}$$

The origin is a spiral source if the eigenvalues are complex and have positive real part

are

A. $b < -\frac{1}{3}$

B. $b > -\frac{1}{3}$

C. $-\frac{1}{4} < b < -\frac{1}{4}$

D. $b > -\frac{1}{4}$

E. $b < -\frac{1}{4}$

Find the eigenvalues:

$$\det \begin{bmatrix} 3-\lambda & b \\ 1 & 4-\lambda \end{bmatrix} = (3-\lambda)(4-\lambda) - b$$

$$= 12 - 3\lambda - 4\lambda + \lambda^2 - b = 0;$$

$$\lambda^2 - 7\lambda + 12 - b = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 4(12-b)}}{2} = \frac{7 \pm \sqrt{1+4b}}{2}.$$

We want $1 + 4b < 0$ $\quad 4b < -1 \quad b < -1/4$

16. Given that $\lambda = 1$ is a defective eigenvalue of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, which of the following is the solution of the initial value problem:

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x} \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}?$$

A. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

B. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2te^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

D D. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

E. $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

We know that $\lambda = 1$ is a defective eigenvalue. The linearly independent solutions are

$X_1(t) = e^t V_1$ and $X_2(t) = e^t (tV_1 + V_2)$

where V_2 is any vector such that $(A-I)V_2 = V_1 \neq 0$

$$A - I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

take $V_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then $V_1 = (A-I)V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $X_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $X_2(t) = e^t \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

$X(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ $\quad X(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ So $c_1 = c_2 = 2$.

$$\boxed{X(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^t \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}$$

20. The general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$

is given by

A. $\mathbf{x}(t) = ae^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} + be^{t}\begin{bmatrix}0\\1\\0\end{bmatrix} + ce^{2t}\begin{bmatrix}1\\2\\1\end{bmatrix}$

B. $\mathbf{x}(t) = ae^{t}\begin{bmatrix}1\\1\\0\end{bmatrix} + be^{-t}\begin{bmatrix}0\\1\\1\end{bmatrix} + ce^{2t}\begin{bmatrix}1\\2\\1\end{bmatrix}$

C. $\mathbf{x}(t) = ae^{t}\begin{bmatrix}1\\0\\1\end{bmatrix} + be^{t}\begin{bmatrix}0\\1\\0\end{bmatrix} + ce^{2t}\begin{bmatrix}1\\2\\0\end{bmatrix}$

D. $\mathbf{x}(t) = ae^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} + be^{t}\begin{bmatrix}1\\1\\0\end{bmatrix} + ce^{2t}\begin{bmatrix}0\\2\\1\end{bmatrix}$

E. $\mathbf{x}(t) = ae^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + be^{2t}\begin{bmatrix}0\\1\\0\end{bmatrix} + ce^{t}\begin{bmatrix}1\\2\\1\end{bmatrix}$

We find the eigenvalues

$$\det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2(2-\lambda)$$

$\lambda = 1$, $\lambda = 2$.

When $\underline{\lambda = 1}$ Find the eigenvector.

$$(A-I)v = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_3 = 0$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = v_1\begin{bmatrix}1\\0\\0\end{bmatrix} + v_2\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\underline{\lambda=2}: \quad (A-2I)v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$-v_1 + v_3 = 0$, $-v_2 + 2v_3 = 0$

$v_1 = v_3$
$v_2 = 2v_3$

$$V = v_1\begin{bmatrix}1\\2\\1\end{bmatrix}$$

General Solution:

$$X(t) = c_1 e^{t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 e^{2t}\begin{bmatrix}1\\2\\1\end{bmatrix}$$